# Getting the Measure Of Inflation

Neil Turok

* Canonical measure on the multiverse
* Scalar field in FRW
* P(N) for inflation
* Answer to Andrei
* Conclusions

Work with Gary Gibbons, hep-th/0609095

The problem of the measure on the set of possible classical space-times is of increasing importance in theoretical cosmology, for example to

* Eternal inflation

* The inflationary multiverse

* Anthropic arguments

* "Top-down" cosmology

Every solution of the classical field equations gives a different space-time: how do we count and compare them?

Gibbons, Hawking & Stewart 1986

Any proposed measure should

i) Count each distinct classical solution once only

ii) Depend only on the intrinsic dynamics and neither on any choice of space-time slicing nor on the choice of fields (e.g. conformal frame)

iii) Respect all of the symmetries of the field equations without introducing any ad hoc cutoffs e.g. the Planck scale

Such a measure is provided by the Hamiltonian structure of general relativity

See also Penrose; Hollands &Wald

# Phase Space Formalities

Canonical 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$$

Constraint sub-manifold
(2n-1 dimensional)

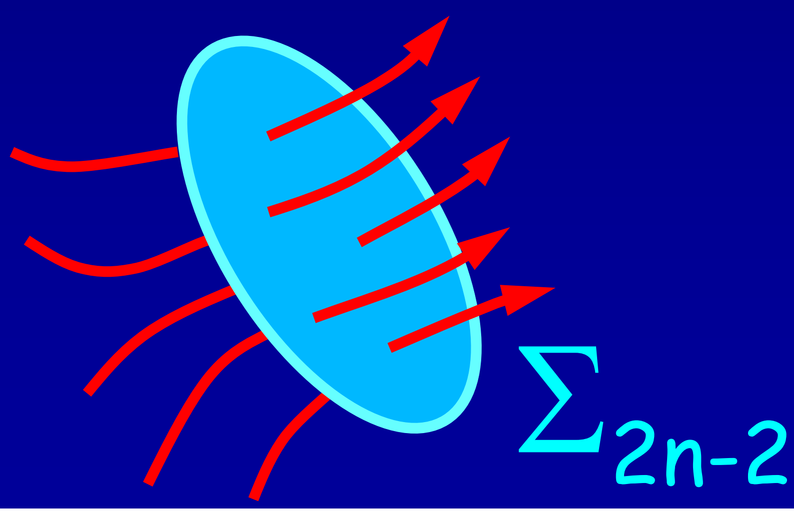
$$C = \mathcal{H}^{-1}(0)$$

Set of distinct classical
solutions (= multiverse)

$$M = C/\mathbb{R} = \mathcal{H}^{-1}(0)/\mathbb{R}$$

$$\omega_C \equiv \omega\big|_{\mathcal{H}=0}$$

$$\Omega_M \equiv \frac{(-1)^{(n-1)(n-2)/2}}{(n-1)!}\omega_C^{n-1}$$



Restricting to constraint manifold C, dw=0 implies that there is a divergence-free "magnetic" field, pointing along the Hamiltonian flow lines: for n=2, we are in three dimensions, and

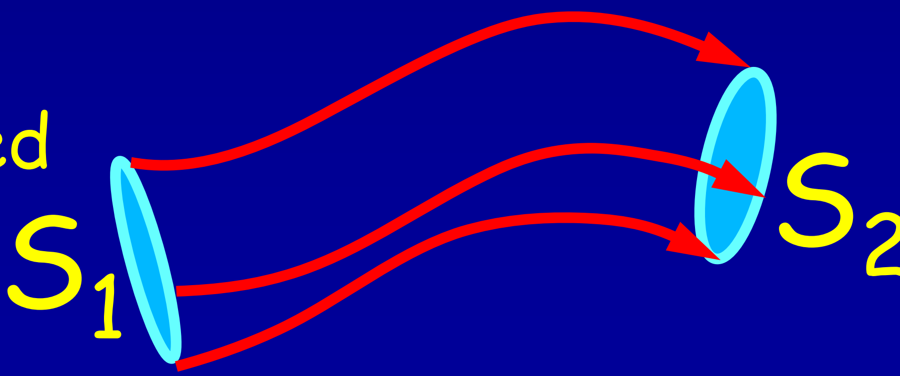
$$B_a \equiv \frac{1}{2}\epsilon_{abc}\omega_{bc}$$

More generally,

$$B_a = \frac{1}{2^{n-1}(n-1)!}\epsilon_{abcde\ldots gh}\omega_{bc}\omega_{de}\ldots\omega_{gh}$$

Flux $\int \mathbf{B}\cdot\mathbf{dS}$ is conserved

This is our measure



Note: we are working in the space of all classical space-times, not in a single space-time with many different regions.

Limitations

* No quantum effects

* Not a "theory of initial conditions": we just count everything

* Homogeneity and isotropy are assumed in present work for simplicity. Anisotropies and inhomogeneities can be included, for finite degrees of freedom

# Scalar field in FRW

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{k}{a^2}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi}$$

~>

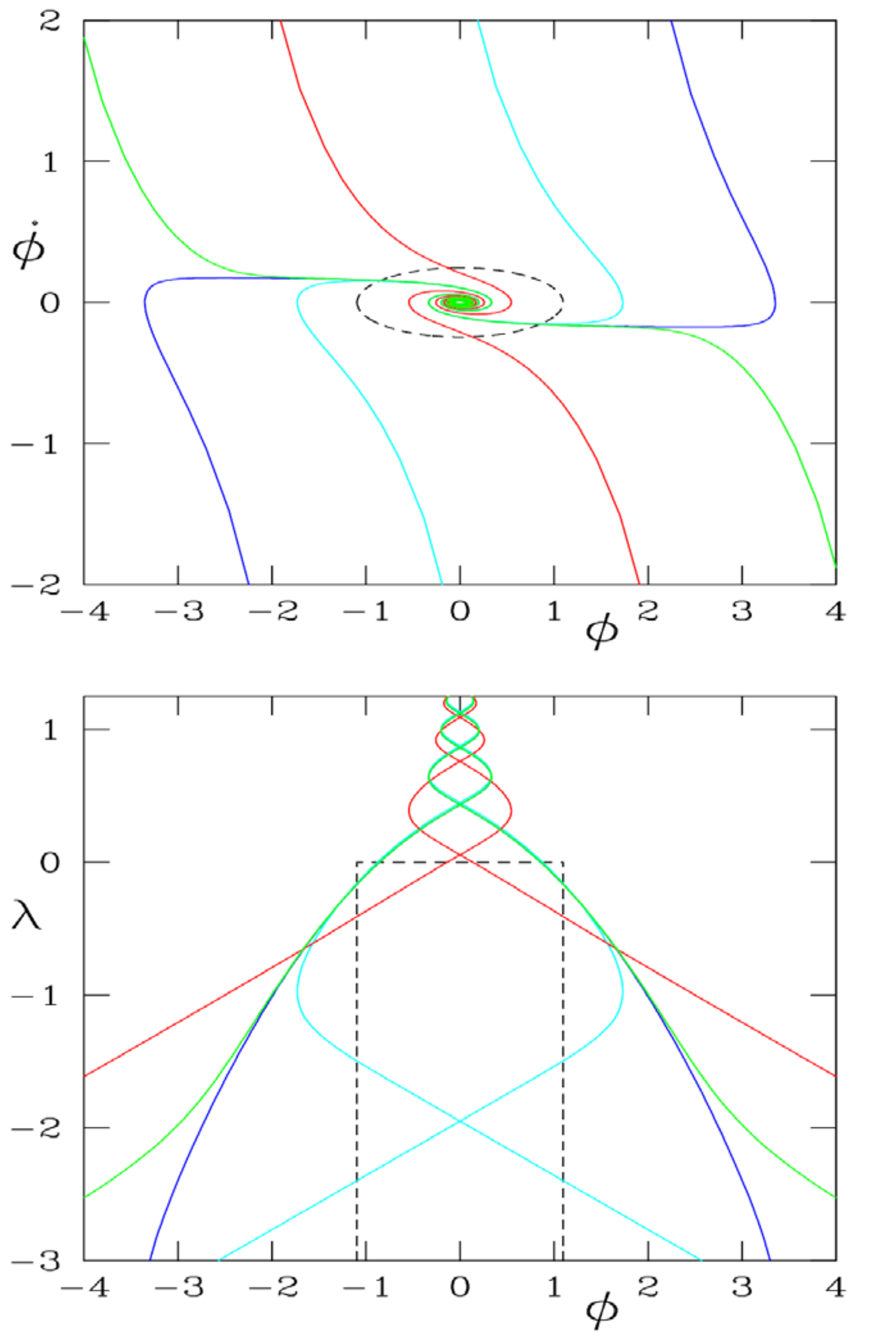
$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 + \frac{k}{a^2}$$

Canonical momenta

$$p_a = -6a\dot{a} = -6a^2 H, \qquad p_\phi = a^3\dot{\phi}$$

Hamiltonian

$$\mathcal{H} = N\left(-\frac{p_a^2}{12a} + \frac{1}{2}\frac{p_\phi^2}{a^3} + a^3 V(\phi) - 3ak\right) = 0$$




# 1) Eliminate $p_a$

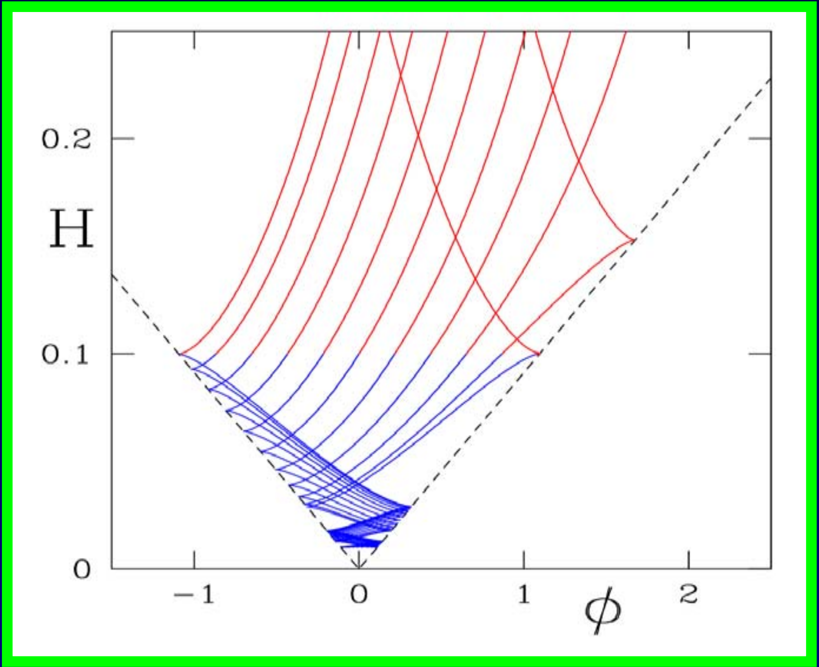
$$(B_\phi, B_{\dot\phi}, B_\lambda) =$$

$$e^{3\lambda}(-\dot\phi/H, 3\dot\phi + V_{,\phi}/H, -1)$$

where $\lambda \equiv \ln a$

Illustration:

$$V(\phi) = \frac{1}{2}m^2\phi^2$$




## 2) Eliminate $p_\phi$

~> remaining coordinates ($\phi$,H,$\lambda$): both $\lambda$ and H monotonic for k=-1,0

$$B_H = \frac{1}{2}\frac{6H^2 - 2V + 4ke^{-2\lambda}}{\sqrt{6H^2 - 2V + 6ke^{-2\lambda}}}$$

Flux through surface
H = $H_S$ = const

$$\int\int 3e^{3\lambda}\frac{6H_S^2 - 2V + 4ke^{-2\lambda}}{\sqrt{6H_S^2 - 2V + 6ke^{-2\lambda}}}\,d\phi d\lambda$$

Diverges (Hawking+Page) at large positive $\lambda$: but in this limit, universe becomes spatially flat and $\lambda$ becomes unobservable. Our proposal is to identify universes with $|\Omega_k| < \Delta\Omega_k$, so that $\lambda$ integral has upper cutoff

$$e^{2\lambda_{max}} = 1/(\Delta\Omega_k H_S^2)$$

Integral is actually topological, since $\omega = d(p\,dq)$ or B = curl A

Flux reduces to

$$2\int a^3|\dot{\phi}|d\phi \equiv \oint p_\phi d\phi$$

i.e. adiabatic invariant for matter field in background FRW, tends to a constant at late times

$$\mathcal{H}_\phi = \frac{1}{2}\frac{p_\phi^2}{a^3(t)} + a^3(t)V(\phi)$$

If $V \sim m^2\phi^2$ around minimum, change$^a$ variables to

"action"

$$\oint p_\phi d\phi = 2\pi m^{-1}\mathcal{H}_\phi \equiv 2\pi J.$$

"angle"

$$\theta = \tan^{-1}\left(\frac{p_\phi}{m q_\phi a^3(t)}\right)$$

Measure indept of $\theta$, depends only on $\rho_\phi a^3$

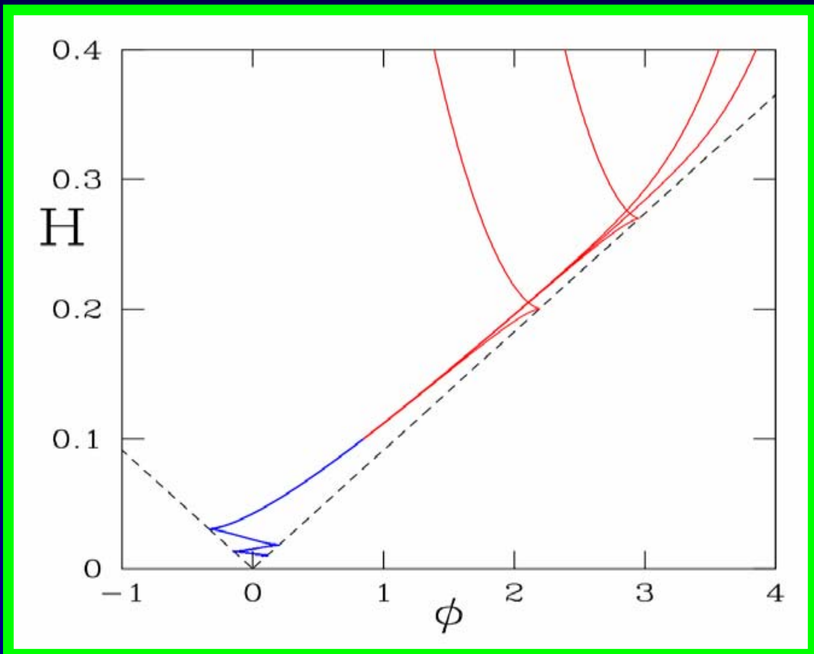
$$\dot{\theta} = -m - \frac{3}{2}\frac{\dot{a}}{a}\sin 2\theta.$$

# Probability for $\Omega_\phi$

$$\int_0^{a_c} d(\rho_\phi a^3) \propto \int_{a_{\min}}^{a_c} d\left(a^3(1+\frac{k}{(aH_S)^2})\right) \propto \int_0^{1+k\Delta\Omega_k} d\left(\frac{\Omega_\phi}{|\Omega_\phi - 1|^{\frac{3}{2}}}\right)$$

From this point of view, it is not such a surprise to find ourselves in a flat universe

# Probability of Inflation




For k=0 SR solution

$$H^2 = \frac{V}{3} + \frac{2}{3}\left(\frac{dH}{d\phi}\right)^2$$

$$H_{SR}(\phi) = \sqrt{\frac{V}{3}}\left(1 + \frac{1}{12}\left(\frac{V_{,\phi}}{V}\right)^2 + \frac{1}{288}\left(-13\left(\frac{V_{,\phi}}{V}\right)^4 + 16\left(\frac{V_{,\phi}}{V}\right)^2\frac{V_{,\phi\phi}}{V}\right.\right.$$
$$\left.\left. + \frac{1}{3456}\left(213\left(\frac{V_{,\phi}}{V}\right)^6 - 432\left(\frac{V_{,\phi}}{V}\right)^4\frac{V_{,\phi\phi}}{V} + 160\left(\frac{V_{,\phi\phi}}{V}\right)^2 + 64\frac{V_{,\phi}}{V}\frac{V_{,\phi\phi\phi}}{V}\right) + \ldots\right)$$

E-folds of expansion

$$\frac{dN}{d\phi} = \frac{H}{\sqrt{6H^2 - 2V}}$$

Deviation from SR

$$\frac{d\delta H}{dN} = 3\delta H$$

(Exact)

Measure for N efolds of slow-roll inflation

$$\int 4 d\phi a^3 \left|\frac{dH}{d\phi}\right|$$

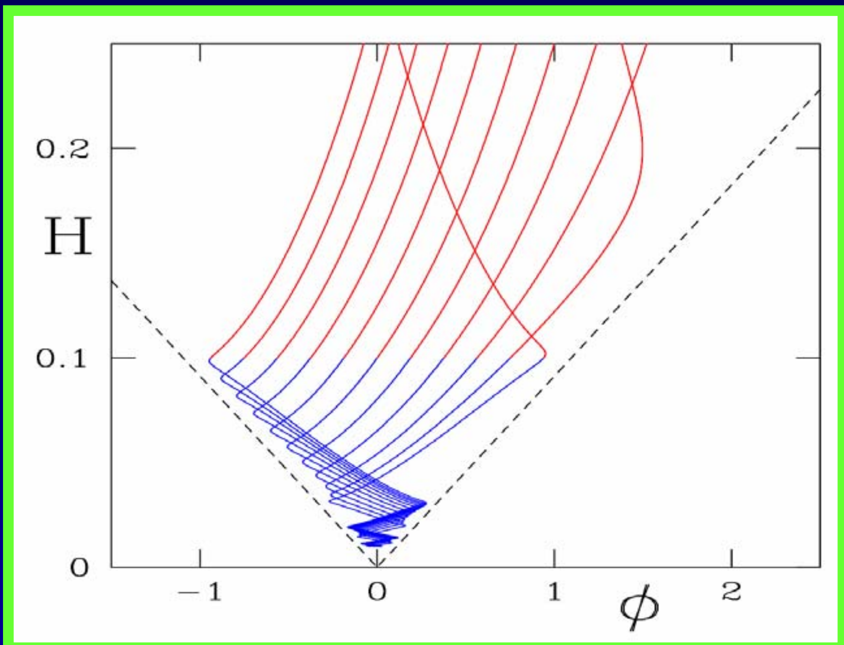
$$\delta H = \delta H_S e^{3N} \approx \left(H - \sqrt{\frac{V}{3}}\right)_{SR}(N) \equiv C(N)$$

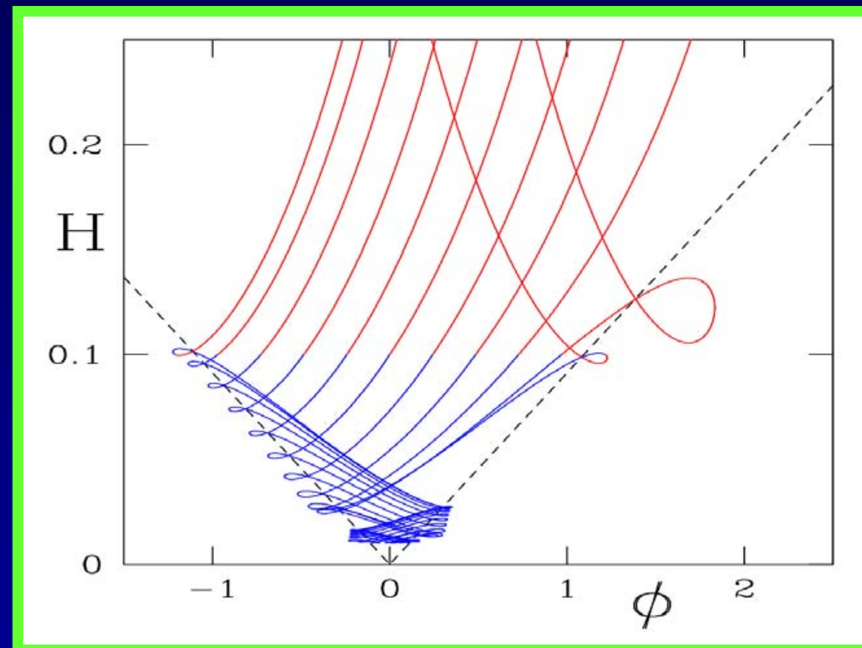
~>

$$P(N) \approx \frac{\delta H_S}{\mathcal{N}} \approx \frac{C(N)e^{-3N}}{\mathcal{N}}, \qquad \text{with} \quad \mathcal{N} \equiv \int_S d\phi\left|\frac{dH}{d\phi}\right|.$$

k=-1

k=+1



# cf. "Planck scale chaos"

Linde;
Kofman, Linde, Mukhanov
…

Trajectories which dominate our measure are kinetic-dominated at early times.

For example, consider $V = m^2 \phi^2/2$, $m \sim 10^{-5}$

Solutions with $N \sim 1$ have $V \sim m^2 \sim 10^{-10}$ when KE hits Planck density.

No reason why equipartition should hold at the Planck density, especially when m has been tuned to be <<1.

# Conclusions

We have developed the canonical measure on the multiverse, which has many good properties.

Obtaining a sensible result required us to identify universes with $\Omega_k$ <<1, which are geometrically and physically indistinguishable. Some probabilities are cutoff-dependent but many are not, e.g. P(N) for inflation.

According to our implementation of the canonical measure, inflationary solutions are exponentially rare in the multiverse. Why the universe we inhabit should be found in one of these is a puzzle for inflationary theory, which requires an explanation.